Notes

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Contents

1 Symbols

1.1 Greek alphabet

A	α	alpha
B	β	beta
Γ	γ	gamma
Δ	δ	delta
E	ϵ	epsilon
Z	ζ	zeta
H	η	eta
Θ	θ	theta
I	ι	iota
K	κ	kappa
Λ	λ	lambda
M	μ	mu
N	ν	nu
Ξ	ξ	xi
O	o	omicron
Π	π	pi
P	ρ	rho
Σ	σ	sigma
T	τ	tau
Υ	υ	upsilon
Φ	ϕ	phi
X	χ	chi
Ψ	ψ	psi
Ω	ω	omega

2 Discrete mathematics

2.1 Logic

2.1.1 Zeroth-order logic

P	$\bot$	$\top$
$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\top$
P	P	$\neg P$
$\bot$	$\bot$	$\top$
$\top$	$\top$	$\bot$

P	Q	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$
$\top$	$\bot$	$\bot$	$\top$
$\top$	$\top$	$\bot$	$\top$

P	Q	$P \wedge Q$	$P \uparrow Q$
$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$
$\top$	$\bot$	$\bot$	$\top$
$\top$	$\top$	$\top$	$\bot$

P	Q	$P \nrightarrow Q$	$P \rightarrow Q$
$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$
$\top$	$\bot$	$\top$	$\bot$
$\top$	$\top$	$\bot$	$\top$

P	Q	P	$\neg P$
$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$
$\top$	$\bot$	$\top$	$\bot$
$\top$	$\top$	$\top$	$\bot$

P	Q	$P \nleftarrow Q$	$P \leftarrow Q$
$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\top$
$\top$	$\top$	$\bot$	$\top$

P	Q	Q	$\neg Q$
$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\top$
$\top$	$\top$	$\top$	$\bot$

P	Q	$P \nleftrightarrow Q$	$P \leftrightarrow Q$
$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\top$	$\bot$
$\top$	$\top$	$\bot$	$\top$

P	Q	$P \vee Q$	$P \downarrow Q$
$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\top$	$\bot$
$\top$	$\top$	$\top$	$\bot$

$$
\begin{aligned}
P \vee P &\iff P \\
P \vee Q &\iff Q \vee P \\
(P \vee Q) \vee R &\iff P \vee (Q \vee R)
\end{aligned}
$$

$$
\begin{aligned}
P \wedge P &\iff P \\
P \wedge Q &\iff Q \wedge P \\
(P \wedge Q) \wedge R &\iff P \wedge (Q \wedge R)
\end{aligned}
$$

$$
\begin{aligned}
P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R) \\
P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)
\end{aligned}
$$

$$
\begin{aligned}
\neg(P \vee Q) &\iff \neg P \wedge \neg Q \\
\neg(P \wedge Q) &\iff \neg P \vee \neg Q
\end{aligned}
$$

2.1.2 First-order logic

$$
\begin{aligned}
\forall x(P(x)) &\iff \neg\exists x(\neg P(x)) \\
\exists x(P(x)) &\iff \neg\forall x(\neg P(x)) \\
\forall x(\forall y(P(x, y))) &\iff \forall y(\forall x(P(x, y))) \\
\exists x(\exists y(P(x, y))) &\iff \exists y(\exists x(P(x, y))) \\
\forall x(P(x)) \wedge \forall x(Q(x)) &\iff \forall x(P(x) \wedge Q(x)) \\
\exists x(P(x)) \vee \exists x(Q(x)) &\iff \exists x(P(x) \vee Q(x))
\end{aligned}
$$

2.2 Set theory

2.2.1 Sets

$$a = b \iff \forall c(c \in a \leftrightarrow c \in b)$$
$$a \subseteq b \iff \forall c(c \in a \rightarrow c \in b)$$

1. axiom of extensionality: $\forall a(\forall b(a = b \leftrightarrow \forall c(a \in c \leftrightarrow b \in c)))$
2. pairing axiom: $\forall a(\forall b(\exists c(\forall d(d \in c \leftrightarrow d = a \vee d = b))))$
3. union axiom: $\forall a(\exists b(\forall c(c \in b \leftrightarrow \exists d(d \in a \wedge c \in d))))$
4. comprehension axiom schema: $\forall a(\exists b(\forall c(c \in b \leftrightarrow c \in a \wedge P(c))))$

$$\bigcup a = \{x \in u : \exists b(b \in a \wedge x \in b)\}$$
$$\bigcap a = \{x \in u : \forall b(b \in a \wedge x \in b)\}$$

$$a' = \{x \in u : x \notin a\}$$
$$a \cup b = \{x \in u : x \in a \vee x \in b\}$$
$$a \cap b = \{x \in u : x \in a \wedge x \in b\}$$
$$a \setminus b = \{x \in u : x \in a \wedge x \notin b\}$$

$$a \cup a = a$$
$$a \cup b = b \cup a$$
$$(a \cup b) \cup c = a \cup (b \cup c)$$

$$a \cap a = a$$
$$a \cap b = b \cap a$$
$$(a \cap b) \cap c = a \cap (b \cap c)$$

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$$
$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$$

$$(a \cup b)' = a' \cap b'$$
$$(a \cap b)' = a' \cup b'$$

5. power set axiom: $\forall a(\exists b(\forall c(c \in b \leftrightarrow c \subseteq a)))$

$$P_a = \{x \in u : x \subseteq a\}$$

6. empty set axiom: $\exists a(\forall b(b \notin a))$
7. axiom of infinity: $\exists a(\emptyset \in a \wedge \forall b(b \in a \rightarrow b \cup \{b\} \in a))$

2.2.2 Binary relations

$$R = (X, Y, G) \qquad G \subseteq X \times Y$$

- total: $\forall x \in X : \exists y \in Y : (x, y) \in G$
- surjective: $\forall y \in Y : \exists x \in X : (x, y) \in G$
- functional: $\forall x \in X : \forall y_1, y_2 \in Y : (x, y_1) \in G \wedge (x, y_2) \in G \rightarrow y_1 = y_2$
- injective: $\forall y \in Y : \forall x_1, x_2 \in X : (x_1, y) \in G \wedge (x_2, y) \in G \rightarrow x_1 = x_2$
- function: total, functional
- bijective function: total, functional, injective, surjective

2.2.3 Functions

$$f : X \rightarrow Y$$
$$x = f^{-1}(y) \mapsto y = f(x)$$

$$f^{-1} : Y \rightarrow X$$
$$y = f(x) \mapsto x = f^{-1}(y)$$

$$f : X \rightarrow Y \qquad g : Y \rightarrow Z$$
$$g \circ f : X \rightarrow Z$$
$$(g \circ f)(x) = g(f(x))$$

$$|X|, |Y| \in \mathbb{N}$$
$$|Y^X| = |Y|^{|X|}$$

$$|X| \in \mathbb{N}$$
$$|S_X| = |X|!$$

2.3 Algebra

2.3.1 Binary relations over a set

$$R = (X, X, G) \qquad G \subseteq X \times X$$

- reflexive: $\forall x \in X : (x, x) \in G$
- symmetric: $\forall x_1, x_2 \in X : (x_1, x_2) \in G \rightarrow (x_2, x_1) \in G$
- antisymmetric: $\forall x_1, x_2 \in X : (x_1, x_2) \in G \wedge (x_2, x_1) \in G \rightarrow x_1 = x_2$
- transitive: $\forall x_1, x_2, x_3 \in X : (x_1, x_2) \in G \wedge (x_2, x_3) \in G \rightarrow (x_1, x_3) \in G$
- equivalence relation: reflexive, symmetric, transitive
- partial order: reflexive, antisymmetric, transitive

2.3.2 Binary operations

$$* : X \times X \rightarrow X$$
$$(x_1, x_2) \mapsto x_1 * x_2$$

- $*$ associative: $\forall x_1, x_2, x_3 \in X : (x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$
- $*$ commutative: $\forall x_1, x_2 \in X : x_1 * x_2 = x_2 * x_1$
- $u_l \in X$ left identity: $\forall x \in X : u_l * x = x$
- $u_r \in X$ right identity: $\forall x \in X : x * u_r = x$
- $u \in X$ identity: $\forall x \in X : u * x = x * u = x$
- $\bar{x}_l \in X$ left inverse of $x \in X$: $\bar{x}_l * x = u$
- $\bar{x}_r \in X$ right inverse of $x \in X$: $x * \bar{x}_r = u$
- $\bar{x} \in X$ inverse of $x \in X$: $\bar{x} * x = x * \bar{x} = u$

2.3.3 Algebraic structures

$(X, *)$ semigroup:

1. $\forall x_1, x_2, x_3 \in X : (x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$

$(X, *)$ monoid:

1. $\forall x_1, x_2, x_3 \in X : (x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$
2. $\exists u \in X : \forall x \in X : u * x = x * u = x$

$(X, *)$ group:

1. $\forall x_1, x_2, x_3 \in X : (x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$
2. $\exists u \in X : \forall x \in X : u * x = x * u = x$
3. $\forall x \in X : \exists \bar{x} \in X : \bar{x} * x = x * \bar{x} = u$

$(X, +, \cdot)$ ring:

- $(X, +)$ Abelian group
  1. $\forall x_1, x_2, x_3 \in X : (x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$
  2. $\exists 0 \in X : \forall x \in X : 0 + x = x + 0 = x$
  3. $\forall x \in X : \exists (-x) \in X : (-x) + x = x + (-x) = 0$
  4. $\forall x_1, x_2 \in X : x_1 + x_2 = x_2 + x_1$
- $(R, \cdot)$ monoid
  1. $\forall x_1, x_2, x_3 \in X : (x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$
  2. $\exists 1 \in X : \forall x \in X : 1 \cdot x = x \cdot 1 = x$
- $\forall x_1, x_2, x_3 \in X : (x_1 + x_2) \cdot x_3 = (x_1 \cdot x_3) + (x_2 \cdot x_3)$
- $\forall x_1, x_2, x_3 \in X : x_1 \cdot (x_2 + x_3) = (x_1 \cdot x_2) + (x_1 \cdot x_3)$

$(X, +, \cdot)$ field:

- $(X, +)$ Abelian group
  1. $\forall x_1, x_2, x_3 \in X : (x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$
  2. $\exists 0 \in X : \forall x \in X : 0 + x = x + 0 = x$
  3. $\forall x \in X : \exists (-x) \in X : (-x) + x = x + (-x) = 0$
  4. $\forall x_1, x_2 \in X : x_1 + x_2 = x_2 + x_1$
- $(R, \cdot)$ Abelian monoid
  1. $\forall x_1, x_2, x_3 \in X : (x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$
  2. $\exists 1 \in X : \forall x \in X : 1 \cdot x = x \cdot 1 = x$
  3. $\forall x_1, x_2 \in X : x_1 \cdot x_2 = x_2 \cdot x_1$
- $\forall x_1, x_2, x_3 \in X : (x_1 + x_2) \cdot x_3 = (x_1 \cdot x_3) + (x_2 \cdot x_3)$
- $\forall x_1, x_2, x_3 \in X : x_1 \cdot (x_2 + x_3) = (x_1 \cdot x_2) + (x_1 \cdot x_3)$
- $\forall x \neq 0 : \exists x^{-1} : x \cdot x^{-1} = x^{-1} \cdot x = 1$

2.3.4 Matrices

$$\det : M_{n \times n}(K) \to K$$

3 Mathematical analysis

3.1 Exponentiation

$$a \in \mathbb{R} \qquad n \in \mathbb{N}$$
$$a^n = \begin{cases} 1 & n = 0 \\ a \cdot a^{n-1} & n \geq 1 \end{cases}$$

$$a \in \mathbb{R} \setminus \{0\} \qquad n \in \mathbb{N}$$
$$a^{-(n+1)} = \frac{1}{a^{n+1}}$$

3.2 Trigonometry

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$
$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = 2\cos^2(x) - 1$$

3.3 Limits

$$x_0 \in \mathbb{R} \qquad \delta \in \{x \in \mathbb{R} : x > 0\}$$
$$I_\delta(x_0) = (x_0 - \delta, x_0 + \delta)$$
$$I_\delta^-(x_0) = (x_0 - \delta, x_0]$$
$$I_\delta^+(x_0) = [x_0, x_0 + \delta)$$

$$\delta \in \mathbb{R}$$
$$I_\delta(+\infty) = (\delta, +\infty)$$
$$I_\delta(-\infty) = (-\infty, \delta)$$

$$x_0 \in \bar{\mathbb{R}} \qquad A \subseteq \mathbb{R}$$
$$x_0 \in D(A) \iff \forall I_\delta(x_0) \in I(x_0) : A \cap I_\delta(x_0) \setminus \{x_0\} \neq \emptyset$$

$$f : D_f \subseteq \mathbb{R} \to \mathbb{R} \qquad x_0 \in D(D_f) \qquad l \in \bar{\mathbb{R}}$$
$$\lim_{x \to x_0} f(x) = l$$
$$\forall I_\epsilon(l) \in I(l) : \exists I_\delta(x_0) \in I(x_0) : \forall x \in I_\delta(x_0) \setminus \{x_0\} : f(x) \in I_\epsilon(l)$$

$$f : D_f \subseteq \mathbb{R} \to \mathbb{R} \qquad x_0 \in D(D_f \cap I_{x_0}(\pm\infty)) \qquad l \in \bar{\mathbb{R}}$$
$$\lim_{x \to x_0^\pm} f(x) = l$$
$$\forall I_\epsilon(l) \in I(l) : \exists I_\delta^\pm(x_0) \in I^\pm(x_0) : \forall x \in I_\delta^\pm(x_0) \setminus \{x_0\} : f(x) \in I_\epsilon(l)$$

4 Physics

4.1 Prefixes

10^{18}	exa	E
10^{15}	peta	P
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10^{1}	deca	da
10^{0}		
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

5 Computer science

5.1 Prefixes

2^{60}	exbi	Ei
2^{50}	pebi	Pi
2^{40}	tebi	Ti
2^{30}	gibi	Gi
2^{20}	mebi	Mi
2^{10}	kibi	Ki

6 Other

6.1 Positional numeral systems

$$b \in \mathbb{N} \setminus \{0, 1\}$$
$$a_0, \ldots, a_m \in \mathbb{R}$$
$$c_1, \ldots, c_n \in \mathbb{R}$$
$$(a_m \cdots a_0.c_1 \cdots c_n)_b = \sum_{i=0}^{m} a_i b^i + \sum_{i=1}^{n} c_i b^{-i}$$

6.2 Natural numbers

$$n \in \mathbb{N}$$
$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

$$n \in \mathbb{N}$$
$$n! = \begin{cases} 1 & n = 0 \\ n(n-1)! & n \geq 1 \end{cases}$$

6.3 Integer numbers

$$a \in \mathbb{Z} \setminus \{0\} \qquad b \in \mathbb{Z}$$
$$a \mid b \iff \exists h \in \mathbb{Z} : b = ha$$

$$a, b \in \mathbb{Z} \qquad n \in \mathbb{N} \setminus \{0\}$$
$$a \equiv b \pmod{n} \iff n \mid (b - a)$$

$$[x]_n = \{y \in \mathbb{Z} : x \equiv y \pmod{n}\}$$

$$\mathbb{Z}_n = \{[0]_n, \ldots, [n-1]_n\}$$

$$a \equiv b \pmod{n} \qquad c \equiv d \pmod{n}$$
$$a + c \equiv b + d \pmod{n}$$
$$ac \equiv bd \pmod{n}$$

$$
\begin{aligned}
\gcd : \mathbb{Z}^2 \setminus \{(0,0)\} &\to \mathbb{N} \setminus \{0\} \\
(z_1, z_2) &\mapsto \gcd(z_1, z_2)
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{lcm} : \mathbb{Z} \setminus \{0\} \times \mathbb{Z} \setminus \{0\} &\to \mathbb{N} \setminus \{0\} \\
(z_1, z_2) &\mapsto \operatorname{lcm}(z_1, z_2)
\end{aligned}
$$

6.4 Real numbers

$$
\begin{aligned}
\sqrt{2} &= 1.4142135623\ldots \\
\sqrt{3} &= 1.7320508075\ldots \\
\sqrt{5} &= 2.2360679774\ldots \\
e &= 2.7182818284\ldots \\
\pi &= 3.1415926535\ldots
\end{aligned}
$$

6.5 Polynomials

$$
\begin{aligned}
p : \mathbb{R} &\to \mathbb{R} \\
p(x) &= \sum_{i=0}^{n} a_i x^i \qquad n \in \mathbb{N}
\end{aligned}
$$

$$
\deg : \mathbb{R}[x] \setminus \{0_{\mathbb{R}[x]}\} \to \mathbb{N}
$$

$$
\begin{aligned}
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x-y)^2 &= x^2 - 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x-y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3 \\
(x+y)(x-y) &= x^2 - y^2 \\
(x+y)(x^2 - xy + y^2) &= x^3 + y^3 \\
(x+y)(x^2 + xy + y^2) &= x^3 - y^3
\end{aligned}
$$

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

$$b^2 - 4ac \geq 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$